Stat 155 Lecture 12 Notes

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1 Indifference of Nash Equilibria, Nash's Theorem, and Potential Games

1.1 Indifference of Nash equilibria in general-sum games

Last lecture, we stated a useful lemma for multiplayer general-sum games.

Lemma 1.1. *Consider a strategy profile $x \in \Delta_{S_1} \times \cdots \times \Delta_{S_k}$. Let $T_i = \{s \in S_i : x_i(s) > 0\}$. Then x is a Nash equilibrium iff for each i there is a c_i such that*

1. *For $s_i \in T_i$, $u_i(s_i, x_{-i}) = c_i$ (indifferent within T_i).*
2. *For $s_i \in S_i$, $u_i(s_i, x_{-i}) \leq c_i$ (no better response outside T_i).*

Proof. ($\implies$) Suppose that x is a Nash equilibrium. Let $i = 1$ and $c_1 := u_1(x)$. Then $u_1(s_1, x_{-1}) \leq u_1(x) = c_1$ for all $s_1 \in S_1$ be the definition of Nash equilibrium. Now observe that

$$
\begin{aligned}
c_1 &= u_1(x) \\
&= \sum_{s_1 \in T_1, s_2 \in S_2, \dots, S_k \in S_k} x_1(s_1) \cdots x_k(s_k) u_1(s_1, \dots, s_k) \\
&= \sum_{s_1 \in T_1} x_1(s_1) \left[\sum_{s_2 \in S_2, \dots, S_k \in S_k} x_2(s_2) \cdots x_k(s_k) u_1(s_1, \dots, s_k) \right] \\
&= \sum_{s_1 \in T_1} x_1(s_1) u_1(s_1, \dots, s_k) \\
&\leq \sum_{s_1 \in T_1} x_1(s_1) u_1(x_1, \dots, s_k) \\
&= \sum_{s_1 \in T_1} x_1(s_1) c_1 \\
&= c_1.
\end{aligned}
$$

Since the inequality is actually an equality, we must have that $u_1(s_1, \dots, s_k) = u_1(x_1, \dots, s_k)$ for each $s_1 \in T_1$.

($\Longleftarrow$) Now assume that the latter conditions hold. Then

$$u_1(x) = u_1(x_1, x_{-1}) = \sum_{s_1 \in T_1} x_1(s_1) u_1(s_1, x_{-1}) = \sum_{s_1 \in T_1} x_1(s_1) c_1 = c_1,$$

and if $\tilde{x} \in \Delta_{S_1}$, then

$$u_1(\tilde{x}_1, x_{-1}) = \sum_{s_1 \in S_1} \tilde{x}_1(s_1) u_1(s_1, x_{-1}) \leq \sum_{s_1 \in S_1} \tilde{x}_1(s_1) c_1 = c_1. \qquad \square$$

1.2 Nash's theorem

Theorem 1.1 (Nash)**.** *Every finite general-sum game has a Nash equilibrium.*

Proof. We give a sketch of the proof for the two player case. We find an "improvement" map $M(x, y) = (\hat{x}, \hat{y})$, so that

1. $\hat{x}^\top A y > x^\top A y$ (or $\hat{x} = x$ if such an $\hat{x}$ does not exist).
2. $x^\top A \hat{y} > x^\top A y$ (or $\hat{y} = y$ if such an $\hat{y}$ does not exist).
3. M is continuous.

A Nash equilibrium is a fixed point of M. The existence of a Nash equilibrium follows from Brouwer's fixed-point theorem.

How do we find M? Set $c_i(x, y) := \max\{e_i^\top A y - x^\top A y, 0\}$. Then define

$$\hat{x}_i = \frac{x_1 + c_i(x, y)}{1 + \sum_{k=1}^m c_k(x, y)}.$$

We can construct $\hat{y}$ in a similar way. $\square$

Here is the precise statement of the theorem that does most of the work in the proof of Nash's theorem.

Theorem 1.2 (Brouwer's Fixed-Point Theorem)**.** *A continuous map $f : K \to K$ from a convex, closed, bounded set $K \subseteq \mathbb{R}^d$ has a fixed point; that is, there exists some $x \in K$ such that $f(x) = x$.*

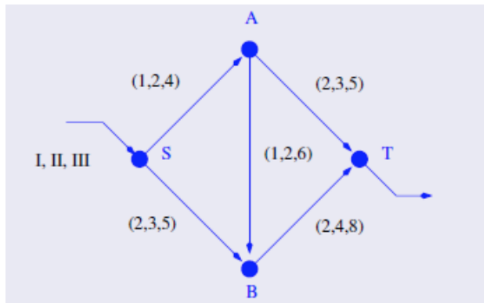
We will not provide a proof, but here is some intuition. In one dimension, a continuous map f from an interval $[a, b]$ to the same interval must intersect the identity map (this is a diagonal of the square $[a, b] \times [a, b]$). In two dimensions, this is related to the Hairy Ball theorem (a hair on a surface must point straight up somewhere). In general, the theorem is non-constructive, so it does not tell us how to get the fixed-point.

Remark 1.1. Not all games have a pure Nash equilibrium. There may only be mixed Nash equilibria.

1.3 Potential and Congestion games

1.3.1 Congestion games

Example 1.1. Consider a game on the following graph:



Three people want to travel from location S to location T and pick a path on the graph. On each of the edges, there is a congestion vector related to how many people choose to take the edge. For example, the edge from B to T takes 2 minutes to traverse if 1 person travels along it, 4 minutes for each person if 2 people travel along it, and 8 minutes for each person if all 3 people travel along the edge. The players each want to minimize the time it takes for them to reach location T.

Definition 1.1. A *congestion game* has k players and m facilities $\{1, \dots, m\}$ (edges). For Player i, there is a set S_i of strategies that are sets of facilities, $s \subseteq \{1, \dots, m\}$ (paths). For facility j, there is a cost vector $c_j \in \mathbb{R}^k$, where $c_j(n)$ is the cost of facility j when it is used by n players.

For a sequence $s = (s_1, \dots, s_n)$, the utilities of the players are defined by

$$\mathrm{cost}_i(s) = -u_i(s) = \sum_{j \in s_i} x_j(n_j(s)),$$

where $n_j(s) = |\{i : j \in s_i\}|$ is the number of players using facility i.

A congestion game is egalitarian in the sense that the utilities depend on how many players use each facility, not on which players use it.

Theorem 1.3. *Every congestion game has a pure Nash equilibrium.*

Proof. We define a potential function $\Phi : S_1 \times \cdots \times S_k \to \mathbb{R}$ as

$$\Phi(s) := \sum_{j=1}^{m} \sum_{\ell=1}^{n_j(s)} c_j(\ell)$$

for fixed strategies for the k players $s = (s_1, \ldots, s_k)$. What happens when Player i changes from s_i to s_i'? We get that

$$\begin{aligned}
\Delta \mathrm{cost}_i &= \mathrm{cost}_i(s_i', s_{-i}) - \mathrm{cost}_i(s) \\
&= \sum_{j \in (s_i', s_{-i})} c_j(n_j(s) + 1) - \sum_{j \in (s_i, s_{-i})} c_j(n_j(s)) \\
&= \Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}) \\
&= \Delta \Phi.
\end{aligned}$$

If we start at an arbitrary s, and update one player's choice to decrease that player's cost, the potential must decrease. Continuing updating other player?s strategies in this way, we must eventually reach a local minimum (there are only finitely many strategies). Since no player can reduce their cost from there, we have reached a pure Nash equilibrium. This gives an algorithm for finding a pure Nash equilibrium: update the choice of one player at a time to reduce their cost. □

1.3.2 Potential games

Definition 1.2. A *potential game* has k players. For Player i, there is a set S_i of strategies and a cost function $\mathrm{cost}_i : S_1 \times \cdots \times S_k \to \mathbb{R}$. A potential game has a *potential function* $\Phi : S_1 \times \cdots \times S_k \to \mathbb{R}$, where

$$\Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}) = \mathrm{cost}_i((s_i', s_{-i}) - \mathrm{cost}_i(s_i, s_{-i}).$$

Congestion games are an example of potential games. In considering congestion games, we actually proved the following theorem.

Theorem 1.4. *Every potential game has a pure Nash equilibrium.*

There is also a converse to the statement that congestion games are potential games.

Theorem 1.5. *Every potential game has an equivalent congestion game.*

Here, an equivalent game means we can find a way to map from the strategies of one game to the strategies of the other so that the utilities are identical. But the congestion game might be much larger: for k players with each $|S_i| = \ell$, the proof involves constructing a congestion game with $2^{k\ell}$ resources.